# Induced subgraphs and subtournaments

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joint with Maria Chudnovsky

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- *(Leaf, S., 2012) For every simple planar graph $H$ with $n$ vertices, taking $k = 2^{O(n \log(n))}$ works.*

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*A minor ideal has bounded $f_2$ if and only if it does not include $I(K_5)$ or $I(K_{3,3})$ ($f_2$ is the minimum $k$ such that deleting some $k$ vertices leaves a graph of genus at most $k$).*

## Theorem

*A minor ideal has*

- *bounded* $f_3$ *iff it does not include the ideal of all graphs that are subgraphs of a path*
- *bounded* $f_4$ *iff it does not include the ideal of all stars*
- *bounded* $f_5$ *iff it does not include the ideal of all graphs with crossing number at most one*
- *bounded* $f_6$ *iff it does not include the ideal of all graphs.*

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Are there any more?

# Tournaments

Possible containment relations:

- subtournament (not wqo)
- topological containment (ie subdivision) (not wqo)
- immersion (wqo)
- butterfly minor (wqo?? – open)
- strong minor (wqo - Kim).

Tournament $G$ has cutwidth at most $c$ if $V(G)$ can be ordered $\{v_1, \dots, v_n\}$ such that for each $i$, there are at most $c$ edges from $\{v_{i+1}, \dots, v_n\}$ to $\{v_1, \dots, v_i\}$.

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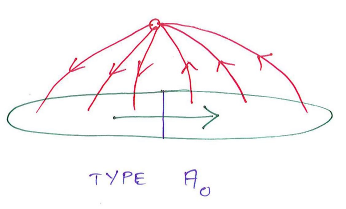
Equivalently:

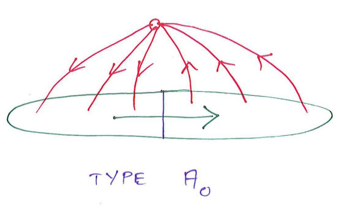
**Theorem**

*An immersion ideal of tournaments has bounded cutwidth iff it does not include {tournaments}.*




**Theorem**

*For every tournament $H$ of type $A_0$, there exists $c$ such that every tournament $G$ not containing $H$ as a subtournament can be ordered such that the maximum backdegree is at most $c$.*

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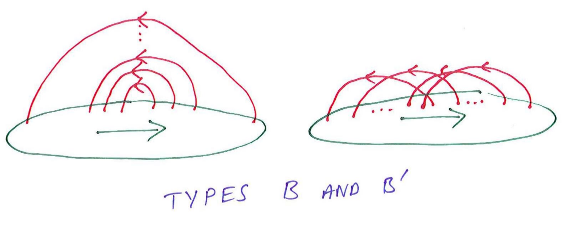
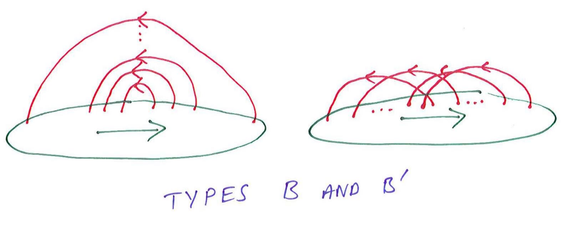
*A subtournament ideal has bounded backdegree iff it does not include {type $A_0$}.*


TYPES B AND B′




## Theorem

*A subtournament ideal has bounded cutwidth iff it does not include {type $A_0$}, {type B} or {type B'}.*

**Theorem (Fradkin, S.)**

*For every tournament $H$ there exists $c$ such that every tournament that does not contain $H$ topologically has pathwidth at most $c$.*

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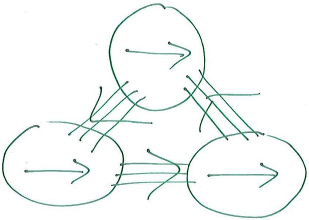
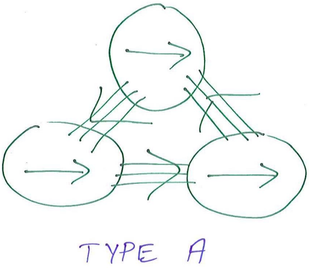
**Theorem (Kim, S.)**

*A strong minor ideal of tournaments has bounded pathwidth iff it does not include {tournaments}.*



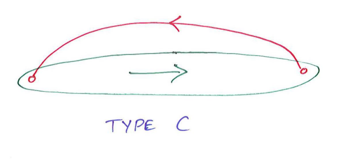
TYPE A




### Theorem

*A subtournament ideal has bounded pathwidth iff it does not include {type A}, {type B} or {type B'}.*

# Tournaments under subtournament containment



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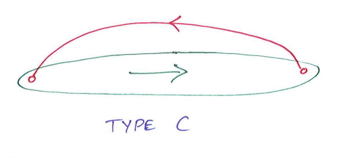
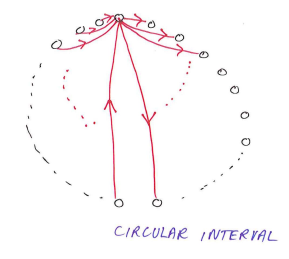


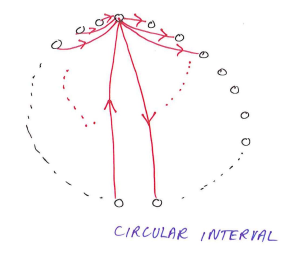

**Theorem**

*A subtournament ideal has bounded length backedges iff it does not include {type C}.*

# Circular interval tournaments

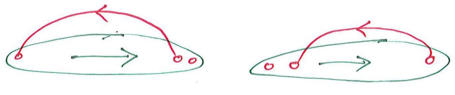


# Circular interval tournaments



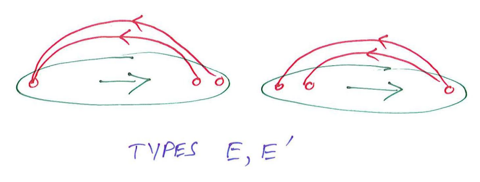

**Theorem**

*A tournament is a circular interval tournament iff it contains neither of the tournaments above.*



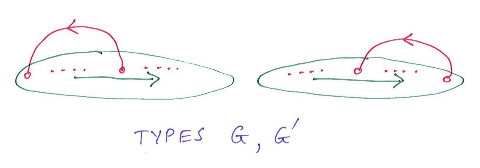

### Theorem (Gaku Liu)

*A subtournament ideal consists of blowups of circular interval tournaments by tournaments with bounded length backedges iff it does not include {type D}, {type D'}.*



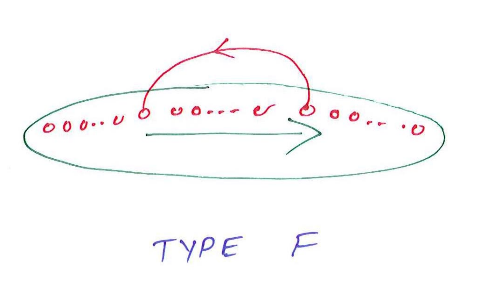

### Theorem

*A subtournament ideal consists of tournaments orderable such that each component of long backedges is a path iff it does not include {type E}, {type E'}.*




**Theorem**

*A subtournament ideal consists of boundedly fuzzy circular interval tournaments iff it does not include {type G}, {type G'}.*




## Problem

A subtournament ideal consists of ??? iff it does not include {type F}.

# Induced subgraphs revisited

A finite set of graphs is heroic if there exists $c$ such that every graph with no induced subgraph in the set has cochromatic number at most $c$.

## Theorem

*Every heroic set contains*

- *a disjoint union of cliques*
- *a complete multipartite graph*
- *a forest*
- *a graph whose complement is a forest.*

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## Conjecture: Gyárfás' 1975; Sumner 1981

For every clique $K$ and forest $F$, there is a constant $c$ such that every graph containing neither of $K, F$ as an induced subgraph has chromatic number at most $c$.

$G$ has splitness at most $k$ if $V(G)$ can be partitioned into $X$, $Y$, where $G|X$ has no clique of size $k+1$ and $G|Y$ has no stable set of size $k+1$.

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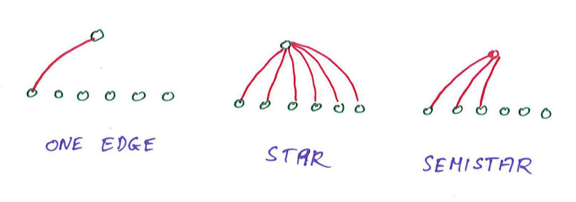
### Corollary

*If Gyarfas' conjecture is true, then every set of graphs containing one of each of the four types is heroic.*




### Theorem

*For every star $H_1$ and star complement $H_2$, there exists $c$ such that if $G$ contains neither of $H_1, H_2$ as an induced subgraph then one of $G, \overline{G}$ has maximum degree at most $c$.*

## Theorem (Reed)

*For every one-edge graph $H_1$ and clique $H_2$, there exists c such that if G contains neither of $H_1, H_2$ as an induced subgraph then G is "almost complete multipartite with at most c parts".*

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## Theorem (Norine, Reed)

*For every semistar $H_1$ and semistar complement $H_2$, there exists a structure theorem.*